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Model Reduction of Nonsquare Linear MIMO Systems Using Multipoint Matrix Continued-fraction Expansions†

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ABSTRACT: *This paper deals with the multipoint Cauer matrix continued-fraction expansion (MCFE) for model reduction of linear multi-input multi-output (MIMO) systems with various numbers of inputs and outputs. A salient feature of the proposed MCFE approach to model reduction of MIMO systems with square transfer matrices is its equivalence to the matrix Padé approximation approach. The Cauer second form of the ordinary MCFE for a square transfer function matrix is generalized in this paper to a multipoint and nonsquare-matrix version. An interesting connection of the multipoint Cauer MCFE method to the multipoint matrix Padé approximation method is established. Also, algorithms for obtaining the reduced-degree matrix-fraction descriptions and reduced-dimensional state-space models from a transfer function matrix via the multipoint Cauer MCFE algorithm are presented. Practical advantages of using the multipoint Cauer MCFE are discussed and a numerical example is provided to illustrate the algorithms.*

I. Introduction

The accurate mathematical modeling or accurate identification of linear time-invariant multi-input multi-output (MIMO) systems usually leads to high-degree

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transfer-function matrices or high-dimensional state-space models. The analysis and design of such high-degree transfer-function matrices in the frequency domain, or such high-dimensional state-space models in the time domain, is not an easy task because of computational difficulties and implementation considerations. There is thus a real incentive to reduce the complexity of linear time-invariant MIMO models to equivalent simple ones.

Several methods are available in the literature (1–6) for approximating a linear time-invariant MIMO system by a reduced-order model. Among them, the method of matrix continued fraction expansion (2–12) has the distinct advantages of computational simplicity and applicability to obtain both frequency-domain as well as time-domain reduced-order models. Chen (2) used a matrix Cauchy second form of continued fractions to obtain a reduced-order model by retaining the first several significant matrix partial quotients and discarding the others. Owing to its equivalence to the matrix Padé approximation about $s = 0$, the method of using the Cauchy second form of MCFE provides satisfactory results in the steady-state response only.

To remove this drawback, Shieh and Gaudiano (9, 10) have proposed an MCFE involving expansion points at $s = 0$ and $s = \infty$, which is equivalent to the matrix Padé approximation about $s = 0$ and $s = \infty$. More recently, Chen and Hwang (7) have presented a multipoint MCFE method to produce better reduced-order models for linear time-invariant MIMO systems.

It is worth noting that the mentioned MCFE methods are derived for the MIMO systems having an equal number of inputs and outputs while being described by their matrix-fraction description (MFD). For broadening the application scope of the MCFE method to include the nonsquare MIMO systems, Shieh *et al.* (11, 13) have proposed the use of a matrix pseudo-inverse for the Cauchy second form of the MCFE. Their method is only applicable to an MIMO system described by its MFD and the connection of the MCFE method to the matrix Padé approximation method has not been explored yet.

The main purpose of this paper is two-fold. Firstly we wish to generalize the CFE algorithm of multipoint Padé fitting (14) to the matrix version for obtaining the reduced-degree MFDs and reduced-dimensional state-space models of a nonsquare linear MIMO system characterized by its transfer-function matrix or its MFD. Secondly we want to reveal the connection of the multipoint Cauchy MCFE method to the multipoint matrix Padé approximation method.

The paper is organized as follows. In Section II, computational algorithms involving a matrix pseudo-inverse are derived for determining the multipoint Cauchy MCFE of a nonsquare linear time-invariant MIMO system from its transfer-function matrix and MFD, respectively. In Section III, an efficient recursive algorithm is presented for obtaining a set of MFDs with different reduced degrees from the multipoint Cauchy MCFE algorithm. Also, the connection of the multipoint Cauchy MCFE method involving the matrix pseudo-inverse to the multipoint matrix Padé approximation method is established. In Section IV, a canonical block state-realization of an MFD based on its multipoint Cauchy MCFE representation is presented for obtaining reduced-dimensional state-space models. An explicit matrix-relationship between the multipoint Cauchy MCFE canonical form and the

block companion form is also derived. Such a matrix-relationship is useful in connection with the construction of an aggregation matrix which relates the state vectors of the system and its reduced-dimensional models.

Throughout this paper, the following notations are used:

- $\mathbf{A}^\#$: left pseudo-inverse of the matrix \mathbf{A} , defined as $(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$;
- \mathbf{A}^\sim : right pseudo-inverse of the matrix \mathbf{A} , defined as $\mathbf{A}^1 (\mathbf{A} \mathbf{A}^1)^{-1}$;
- \mathbf{I}_n : identity matrix of order n ;
- \mathbf{O}_n : null matrix of order n ;
- $\mathbf{O}_{p \times q}$: p by q null matrix;
- $\mathbf{R}^{p \times q}$: set of $p \times q$ real matrices.

II. Multipoint Cauer MCFE of a Nonsquare Transfer-function Matrix

A linear time-invariant q -input p -output system may be described by the $p \times q$ transfer-function matrix

$$\mathbf{G}(s) = [g_{i,j}(s)], \quad i = 1, 2, \dots, p; \quad j = 1, 2, \dots, q \quad (1)$$

or by the matrix-fraction description (MFD) (15):

$$\mathbf{G}(s) = \left[\sum_{i=0}^{N-1} \mathbf{A}_{2,i} s^i \right] \left[\sum_{i=0}^{N-1} \mathbf{A}_{1,i} s^i \right]^{-1} \triangleq \mathbf{A}_2(s) \mathbf{A}_1(s)^{-1}, \quad p \geq q \quad (2)$$

where $\mathbf{A}_{1,i} \in \mathbf{R}^{q \times q}$ and $\mathbf{A}_{2,i} \in \mathbf{R}^{p \times q}$. Let the Taylor series expansions of $\mathbf{G}(s)$ about the m distinct real points $\sigma_1, \sigma_2, \dots, \sigma_m$, be

$$\mathbf{G}(s) = \sum_{j=0}^i \mathbf{T}_{i,j} (s - \sigma_i)^j, \quad i = 1, 2, \dots, m \quad (3a)$$

where

$$\mathbf{T}_{i,j} = \frac{1}{j!} \frac{d^j}{ds^j} \mathbf{G}(s) \Big|_{s=\sigma_i} \in \mathbf{R}^{p \times q}. \quad (3b)$$

In the following, we deal with the MCFE for the nonsquare transfer-function matrix $\mathbf{G}(s)$. Before proceeding, let $\mu_1, \mu_2, \dots, \mu_m$ be the multiplicities corresponding to the distinct real points $\sigma_1, \sigma_2, \dots, \sigma_m$, such that $\mu_1 + \mu_2 + \dots + \mu_m = 2N$. These numbers σ_i will be used as expansion points taking into account multiplicities. Let

$$(s_1, s_2, \dots, s_{2N}) = (\underbrace{\sigma_1, \dots, \sigma_1}_{\mu_1}, \underbrace{\sigma_2, \dots, \sigma_2}_{\mu_2}, \dots, \underbrace{\sigma_m, \dots, \sigma_m}_{\mu_m})$$

denote the ordered point system. Also, let

$$v_i = \mu_1 + \mu_2 + \dots + \mu_{i-1}. \quad (4)$$

2.1. Multipoint Cauer MCFE of an MFD

From the definition of a matrix pseudo-inverse and the MFD (2), we can write $\mathbf{G}(s)$ as

$$\mathbf{G}(s) = [\mathbf{G}^\#(s)]^\sim = [[\mathbf{A}_2(s) \mathbf{A}_1^{-1}(s)]^\#]^\sim = [\mathbf{A}_1(s) \mathbf{A}_2^\#(s)]^\sim. \quad (5)$$

Assume that $\mathbf{A}_1^\#(s_1)$ exists and let

$$\mathbf{H}_1 = \mathbf{A}_1(s_1)\mathbf{A}_2^\#(s_1) \in R^{q \times p}. \quad (6)$$

Then, we can expand $\mathbf{A}_1(s)\mathbf{A}_2^\#(s)$ as

$$\mathbf{A}_1(s)\mathbf{A}_2^\#(s) = \mathbf{H}_1 + (s-s_1)\mathbf{A}_3(s)\mathbf{A}_2^\#(s) \quad (7)$$

where the matrix polynomial

$$\mathbf{A}_3(s) = \sum_{j=0}^{N-1} \mathbf{A}_{3,j}s^j, \quad \mathbf{A}_{3,j} \in R^{q \times q} \quad (8)$$

is to be determined. Postmultiplying both sides of (7) by $\mathbf{A}_2(s)$ and then using the identity $\mathbf{A}_2^\#(s)\mathbf{A}_2(s) = \mathbf{I}_q$ yields the relation:

$$\mathbf{A}_1(s) = \mathbf{H}_1\mathbf{A}_2(s) + (s-s_1)\mathbf{A}_3(s). \quad (9)$$

Substituting the matrix polynomials of $\mathbf{A}_1(s)$, $\mathbf{A}_2(s)$ and $\mathbf{A}_3(s)$ into (9), and then equating the coefficients of like powers, we obtain

$$\mathbf{A}_{3,j} = \mathbf{A}_{1,j+1} - \mathbf{H}_1\mathbf{A}_{2,j+1} + s_1\mathbf{A}_{3,j+1}, \quad j = N-1, N-2, \dots, 0 \quad (10)$$

where $\mathbf{A}_{2,N} = \mathbf{O}_{p \times q}$ and $\mathbf{A}_{3,N} = \mathbf{O}_q$.

With the substitution of (7) into (5), $\mathbf{G}(s)$ can be further written as

$$\mathbf{G}(s) = [\mathbf{H}_1 + (s-s_1)\mathbf{A}_3(s)\mathbf{A}_2^\#(s)]^\sim = [\mathbf{H}_1 + (s-s_1)[\mathbf{A}_2(s)\mathbf{A}_3^{-1}(s)]^\#]^\sim. \quad (11)$$

Similarly, assume that $\mathbf{A}_3^{-1}(s_2)$ exists, and let

$$\mathbf{H}_2 = \mathbf{A}_2(s_2)\mathbf{A}_3^{-1}(s_2) \in R^{p \times q} \quad (12)$$

then the term in the inner brackets of (11) can be expanded as

$$\mathbf{A}_2(s)\mathbf{A}_3^{-1}(s) = \mathbf{H}_2 + (s-s_2)\mathbf{A}_4(s)\mathbf{A}_3^{-1}(s) \quad (13)$$

where the matrix polynomial

$$\mathbf{A}_4(s) = \sum_{j=0}^{N-2} \mathbf{A}_{4,j}s^j, \quad \mathbf{A}_{4,j} \in R^{p \times q} \quad (14)$$

satisfies the relation:

$$\mathbf{A}_2(s) = \mathbf{H}_2\mathbf{A}_3(s) + (s-s_2)\mathbf{A}_4(s). \quad (15)$$

From this we have

$$\mathbf{A}_{4,j} = \mathbf{A}_{2,j+1} - \mathbf{H}_2\mathbf{A}_{3,j+1} + s_2\mathbf{A}_{4,j+1}, \quad j = N-2, N-1, \dots, 0 \quad (16)$$

where $\mathbf{A}_{4,j} = \mathbf{O}_{p \times q}$ for $j \geq N-1$. With the substitution of (13) into (11), $\mathbf{G}(s)$ is now expanded as

$$\mathbf{G}(s) = [\mathbf{H}_1 + (s-s_1)[\mathbf{H}_2 + (s-s_2)\mathbf{A}_4(s)\mathbf{A}_3^{-1}(s)]^\#]^\sim. \quad (17)$$

By repeating the expansion procedure of (5)–(17) in the MFD, $\mathbf{A}_4(s)\mathbf{A}_3^{-1}(s)$, we can finally expand $\mathbf{G}(s)$ into the following multipoint Cauchy matrix continued

with

$$\begin{aligned} \mathbf{G}_{i,1}(s) &= \mathbf{I}_q + \mathbf{O}_q(s - \sigma_i) + \mathbf{O}_q(s - \sigma_i)^2 + \dots \\ &\triangleq \sum_{k=0}^{\infty} \mathbf{G}_{i,1,k}(s - \sigma_i)^k, \quad i = 1, 2, \dots, m \end{aligned} \quad (20)$$

and

$$\begin{aligned} \mathbf{G}_{i,2}(s) &= \mathbf{T}_{i,0} + \mathbf{T}_{i,1}(s - \sigma_i) + \mathbf{T}_{i,2}(s - \sigma_i)^2 + \dots \\ &\triangleq \sum_{k=0}^{\infty} \mathbf{G}_{i,2,k}(s - \sigma_i)^k, \quad i = 1, 2, \dots, m. \end{aligned} \quad (21)$$

Then, from (3a), (20) and (21), we can represent $\mathbf{G}(s)$ by one of the following infinite-term MFDs:

$$\begin{aligned} \mathbf{G}(s) &= \mathbf{G}_{i,2}(s)\mathbf{G}_{i,1}^{-1}(s) \\ &= [\mathbf{G}_{i,1}(s)\mathbf{G}_{i,2}^{\#}(s)]^{\sim}, \quad i = 1, 2, \dots, m. \end{aligned} \quad (22)$$

According to the form of the multipoint Caer MCFE (18) and the expansion ordered point system (4), we know that matrix partial quotients $\mathbf{H}_{v_i+1}, \mathbf{H}_{v_i+2}, \dots, \mathbf{H}_{v_i+\mu_i}$ are evaluated from the infinite-term MFD $\mathbf{G}_{i,2}(s)\mathbf{G}_{i,1}^{-1}(s)$. Since the multipoint Caer MCFE involves m distinct expansion points $\sigma_i, i = 1, 2, \dots, m$, it is therefore required to manipulate the m infinite-term MFDs in (22), which represent the same transfer-function matrix $\mathbf{G}(s)$, to obtain the partial quotients \mathbf{H}_i of the multipoint Caer MCFE (18).

Now, starting from (22), we expand the infinite-term MFD $\mathbf{G}_{i,1}(s)\mathbf{G}_{i,2}^{\#}(s)$ as

$$\mathbf{G}_{i,1}(s)\mathbf{G}_{i,2}^{\#}(s) = \mathbf{H}_1 + (s - s_1)\mathbf{G}_{i,3}(s)\mathbf{G}_{i,2}^{\#}(s), \quad i = 1, 2, \dots, m \quad (23)$$

where

$$\begin{aligned} \mathbf{H}_1 &= \mathbf{G}_{i,1}(s_1)\mathbf{G}_{i,2}^{\#}(s_1) \quad \text{for } s_1 = \sigma_1 \\ &= \mathbf{G}_{i,1,0}\mathbf{G}_{i,2,0}^{\#}. \end{aligned} \quad (24)$$

In (23), each of the infinite-term series $\mathbf{G}_{i,3}(s)$ satisfies

$$\mathbf{G}_{i,1}(s) = \mathbf{H}_1\mathbf{G}_{i,2}(s) + (s - s_1)\mathbf{G}_{i,3}(s), \quad i = 1, 2, \dots, m \quad (25)$$

which is obtained by postmultiplying both sides of (23) with $\mathbf{G}_{i,2}(s)$ and using the identity $\mathbf{G}_{i,2}^{\#}(s)\mathbf{G}_{i,2}(s) = \mathbf{I}_q$. Substituting the series representations of $\mathbf{G}_{i,1}(s)$, $\mathbf{G}_{i,2}(s)$ and $\mathbf{G}_{i,3}(s)$ into (25), we have

$$\begin{aligned} \sum_{k=0}^{\infty} \mathbf{G}_{i,1,k}(s - \sigma_i)^k &= \mathbf{H}_1 \sum_{k=0}^{\infty} \mathbf{G}_{i,2,k}(s - \sigma_i)^k \\ &\quad + [(s - \sigma_i) + (\sigma_i - s_1)] \sum_{k=0}^{\infty} \mathbf{G}_{i,3,k}(s - \sigma_i)^k. \end{aligned} \quad (26)$$

Equating the coefficients of like terms $(s - \sigma_i)^k$, we obtain

$$\mathbf{G}_{i,3,k} = \begin{cases} \frac{1}{\sigma_i - s_1} [\mathbf{G}_{i,1,k} - \mathbf{H}_1 \mathbf{G}_{i,2,k} - \mathbf{G}_{i,3,k-1}], & i = 2, 3, \dots, m; k = 0, 1, 2, \dots \\ \mathbf{G}_{i,1,k+1} - \mathbf{H}_1 \mathbf{G}_{i,2,k+1}, & i = 1; k = 0, 1, 2, \dots \end{cases} \quad (27)$$

where $\mathbf{G}_{i,3,-1} = \mathbf{O}_q$.

In general, the MFDs, $\mathbf{G}_{i,j}(s)\mathbf{G}_{i,j+1}^\#(s)$ for j odd and $\mathbf{G}_{i,j}(s)\mathbf{G}_{i,j+1}^1(s)$ for j even, can respectively be expanded as

$$\mathbf{G}_{i,j}(s)\mathbf{G}_{i,j+1}^\#(s) = \mathbf{H}_j + (s - s_j)\mathbf{G}_{i,j+2}(s)\mathbf{G}_{i,j+1}^\#(s), \quad i = l, l+1, \dots, m \quad (28a)$$

and

$$\mathbf{G}_{i,j}(s)\mathbf{G}_{i,j+1}^1(s) = \mathbf{H}_j + (s - s_j)\mathbf{G}_{i,j+2}(s)\mathbf{G}_{i,j+1}^1(s), \quad i = l, l+1, \dots, m \quad (28b)$$

where $s_j = \sigma_j$ and

$$\mathbf{H}_j = \begin{cases} \mathbf{G}_{l,j}(\sigma_j)\mathbf{G}_{l,j+1}^\#(\sigma_j) = \mathbf{G}_{l,j,0}\mathbf{G}_{l,j+1,0}^\# & \text{for } j \text{ odd} \\ \mathbf{G}_{l,j}(\sigma_j)\mathbf{G}_{l,j+1}^1(\sigma_j) = \mathbf{G}_{l,j,0}\mathbf{G}_{l,j+1,0}^1 & \text{for } j \text{ even.} \end{cases} \quad (29)$$

In (28), each infinite-term series $\mathbf{G}_{i,j+2}(s)$ satisfies

$$\mathbf{G}_{i,j}(s) = \mathbf{H}_j \mathbf{G}_{i,j+1}(s) + (s - s_j)\mathbf{G}_{i,j+2}(s), \quad i = l, l+1, \dots, m \quad (30a)$$

i.e.

$$\sum_{k=0}^l \mathbf{G}_{i,j,k}(s - \sigma_i)^k = \mathbf{H}_j \sum_{k=0}^l \mathbf{G}_{i,j+1,k}(s - \sigma_i)^k + [(s - \sigma_i) + (\sigma_i - s_j)] \sum_{k=0}^l \mathbf{G}_{i,j+2,k}(s - \sigma_i)^k. \quad (30b)$$

This last equation allows one to evaluate the coefficients $\mathbf{G}_{i,j+2,k}$ for $i = l, l+1, \dots, m$ and $k = 0, 1, 2, \dots$, as follows:

$$\mathbf{G}_{i,j+2,k} = \begin{cases} \frac{1}{\sigma_i - s_j} [\mathbf{G}_{i,j,k} - \mathbf{H}_j \mathbf{G}_{i,j+1,k} - \mathbf{G}_{i,j+2,k-1}], & i = l+1, l+2, \dots, m \\ \mathbf{G}_{i,j,k+1} - \mathbf{H}_j \mathbf{G}_{i,j+1,k+1}, & i = l. \end{cases} \quad (31)$$

Following the above derivation, we can construct m expansion arrays as shown in Table II for calculating the matrix partial quotients $\mathbf{H}_1, \mathbf{H}_2, \dots, \mathbf{H}_{2N}$ of the multipoint Caer MCFE of $\mathbf{G}(s)$ from the matrix coefficients of the Taylor series expansions in (3).

In the expansion array in Table II, the first two matrix rows are specified by

$$\mathbf{G}_{i,1,k} = \begin{cases} \mathbf{I}_q & \text{for } k = 0 \\ \mathbf{O}_q & \text{for } k = 1, 2, \dots, \mu_i - 1 \end{cases} \quad (32a)$$

$$\mathbf{G}_{i,2,k} = \mathbf{T}_{i,k} \quad \text{for } k = 0, 1, \dots, \mu_i - 1. \quad (32b)$$

TABLE II
Expansion array for evaluating $G_{i,j,k}$

		$G_{i,1,0}$	$G_{i,1,1}$	$G_{i,1,2}$...	$G_{i,1,\mu_i-2}$	$G_{i,1,\mu_i-1}$
		$G_{i,2,0}$	$G_{i,2,1}$	$G_{i,2,2}$...	$G_{i,2,\mu_i-2}$	$G_{i,2,\mu_i-1}$
		$G_{i,3,0}$	$G_{i,3,1}$	$G_{i,3,2}$...	$G_{i,3,\mu_i-2}$	$G_{i,3,\mu_i-1}$
		\vdots					
H_{v_i+1}	<	$G_{i,v_i+1,0}$	$G_{i,v_i+1,1}$	$G_{i,v_i+1,2}$...	G_{i,v_i+1,μ_i-2}	G_{i,v_i+1,μ_i-1}
H_{v_i+2}	<	$G_{i,v_i+2,0}$	$G_{i,v_i+2,1}$	$G_{i,v_i+2,2}$...	G_{i,v_i+2,μ_i-2}	G_{i,v_i+2,μ_i-1}
	<	$G_{i,v_i+3,0}$	$G_{i,v_i+3,1}$	$G_{i,v_i+3,2}$...	G_{i,v_i+3,μ_i-2}	
		\vdots					
$H_{v_i+\mu_i-1}$	<	$G_{i,v_i+\mu_i-1,0}$	$G_{i,v_i+\mu_i-1,1}$	$G_{i,v_i+\mu_i-1,2}$			
$H_{v_i+\mu_i}$	<	$G_{i,v_i+\mu_i,0}$	$G_{i,v_i+\mu_i,1}$				
	<	$G_{i,v_i+\mu_i+1,0}$					

The subsequent matrix rows of the expansion array are obtained by the following recurrence formulas,

$$G_{i,j+2,k} = \begin{cases} \frac{1}{\sigma_i - s_j} [G_{i,j,k} - H_j G_{i,j+1,k} - G_{i,j+2,k-1}], \\ \quad k = 0, 1, \dots, \mu_i - 1 \text{ and } j = 1, 2, \dots, v_i \\ G_{i,j,k+1} - H_j G_{i,j+1,k+1}, \\ \quad k = 0, 1, \dots, v_i + \mu_i - j - 1 \text{ and } j = v_i + 1, v_i + 2, \dots, v_i + \mu_i - 1 \end{cases} \quad (33)$$

which are derived from the relation in (30). The matrix partial quotients H_j for $j = v_i + 1, v_i + 2, \dots, v_i + \mu_i$ associated with the expansion array in Table II are evaluated as

$$H_j = \begin{cases} G_{i,j,0} G_{i,j+1,0}^\# & \text{for } j \text{ odd} \\ G_{i,j,0} G_{i,j+1,0}^{-1} & \text{for } j \text{ even.} \end{cases} \quad (34)$$

It is noted that in the foregoing derivations, we have assumed that the required matrix inverse and matrix pseudo-inverse exist. If $G_{i,j,0}^{-1}$ or $G_{i,j,0}^\#$ for $i = 1, 2, \dots, m$ do not exist, we should readjust the expansion points σ_i such that the matrix inverse exists. When all coefficients of any matrix row in the expansion array are null, the expansion procedure terminates. In this case, we can use the calculated matrix partial quotients and the inversion procedure in the next section to derive the MFD for $G(s)$.

III. Reduced-degree MFDs Derived from the Multipoint Cauer MCFE

Suppose the transfer-function matrix $G(s)$ of a high-degree q -input p -output system is expanded into a multipoint Cauer MCFE in (18). An M th-degree reduced MFD, $\hat{G}_M(s)$, for the system can be obtained by truncating the matrix partial

and the remaining matrix entries are evaluated using the following recurrence relations:

$$\mathbf{P}_{k,j} = \mathbf{P}_{k-1,j} \mathbf{H}_k - s_{k-1} \mathbf{P}_{k-2,j} + \mathbf{P}_{k-2,j-1}, \quad j = 0, 1, \dots, [(k-1)/2] \quad (40a)$$

$$\mathbf{Q}_{k,j} = \mathbf{Q}_{k-1,j} \mathbf{H}_k - s_{k-1} \mathbf{Q}_{k-2,j} + \mathbf{Q}_{k-2,j-1}, \quad j = 0, 1, \dots, [k/2] \quad (40b)$$

where

$$\mathbf{P}_{k,-1} = \begin{cases} \mathbf{O}_{p \times q} & \text{for } k \text{ even} \\ \mathbf{O}_p & \text{for } k \text{ odd} \end{cases}$$

and

$$\mathbf{Q}_{k,-1} = \begin{cases} \mathbf{Q}_q & \text{for } k \text{ even} \\ \mathbf{O}_{q \times p} & \text{for } k \text{ odd} \end{cases}$$

for $k = 1, 2, \dots, M$.

Once the inversion array is developed, the M th-degree reduced MFD, which corresponds to the truncated MCFE in (35), can be readily written as

$$\begin{aligned} \hat{\mathbf{G}}_M(s) &= \mathbf{P}_{2M}(s) \mathbf{Q}_{2M}^{-1}(s) \\ &= [\mathbf{P}_{2M,0} + \mathbf{P}_{2M,1}s + \dots + \mathbf{P}_{2M,M-1}s^{M-1}] [\mathbf{Q}_{2M,0} \\ &\quad + \mathbf{Q}_{2M,1}s + \dots + \mathbf{Q}_{2M,M}s^M]^{-1}. \end{aligned} \quad (41)$$

It is important here to examine the link between the M th-degree reduced MFD, $\hat{\mathbf{G}}_M(s)$ and the original transfer-function matrix $\mathbf{G}(s)$. As a specific example, we consider $\hat{\mathbf{G}}_1(s)$ which has the following explicit form:

$$\hat{\mathbf{G}}_1(s) = \mathbf{H}_2 [\mathbf{H}_1 \mathbf{H}_2 + (s - s_1) \mathbf{I}_q]^{-1}. \quad (42)$$

With \mathbf{H}_i being represented by (34), we have the following explicit expressions for $\hat{\mathbf{G}}_1(s_1)$, $\hat{\mathbf{G}}_1(s_2)$ and $(d/ds)\hat{\mathbf{G}}_1(s_1)$:

$$\begin{aligned} \hat{\mathbf{G}}_1(s_1) &= \mathbf{H}_2 [\mathbf{H}_1 \mathbf{H}_2]^{-1} \\ &= \begin{cases} \mathbf{G}_{1,2,0} \mathbf{G}_{1,3,0}^{-1} [\mathbf{G}_{1,1,0} \mathbf{G}_{1,2,0}^{\#} \mathbf{G}_{1,2,0} \mathbf{G}_{1,3,0}]^{-1} & \text{for } s_2 = s_1 \\ \mathbf{G}_{2,2,0} \mathbf{G}_{2,3,0}^{-1} [\mathbf{G}_{1,1,0} \mathbf{G}_{1,2,0}^{\#} \mathbf{G}_{2,2,0} \mathbf{G}_{2,3,0}]^{-1} & \text{for } s_2 \neq s_1 \end{cases} \end{aligned} \quad (43a)$$

$$\begin{aligned} \hat{\mathbf{G}}_1(s_2) &= \mathbf{H}_2 [\mathbf{H}_1 \mathbf{H}_2 + (s_2 - s_1) \mathbf{I}_q]^{-1} \\ &= \mathbf{G}_{2,2,0} [\mathbf{H}_1 \mathbf{G}_{2,2,0} - (s_2 - s_1) \mathbf{G}_{2,3,0}]^{-1} \\ &= \mathbf{G}_{2,2,0} \mathbf{G}_{2,1,0}^{-1} = \mathbf{G}_{2,2,0} \end{aligned} \quad (43b)$$

$$\begin{aligned} \frac{d}{ds} \hat{\mathbf{G}}_1(s_1) &= -\mathbf{H}_2 [(\mathbf{H}_1 \mathbf{H}_2)^2]^{-1} \\ &= \begin{cases} -\mathbf{G}_{1,2,0} \mathbf{G}_{1,3,0} = \mathbf{G}_{1,2,0} \mathbf{G}_{1,2,0}^{\#} \mathbf{G}_{1,2,1} & \text{for } s_2 = s_1 \\ -\mathbf{G}_{2,2,0} \mathbf{G}_{2,3,0} = [(\mathbf{G}_{1,2,0}^{\#} \mathbf{G}_{2,2,0})^2]^{-1} & \text{for } s_2 \neq s_1. \end{cases} \end{aligned} \quad (43c)$$

Comparing (43) and (21), we have

$$\hat{\mathbf{G}}_1(s_1) = \mathbf{G}(s_1), \quad \hat{\mathbf{G}}_1(s_2) = \mathbf{G}(s_2) \quad \text{for } s_1 \neq s_2 \text{ and } p = q \quad (44a)$$

$$\hat{\mathbf{G}}_1(s_1) = \mathbf{G}(s_1), \quad \frac{d}{ds} \hat{\mathbf{G}}_1(s_2) = \frac{d}{ds} \mathbf{G}(s_2) \quad \text{for } s_1 = s_2 \text{ and } p = q \quad (44b)$$

$$\hat{\mathbf{G}}_1(s_1) \neq \mathbf{G}(s_1), \quad \hat{\mathbf{G}}_1(s_2) = \mathbf{G}(s_2) \quad \text{for } p \neq q. \quad (44c)$$

By a similar direct deduction, we can obtain the link between the M th-degree ($M < N$) reduced MFD, $\hat{\mathbf{G}}_M(s)$, in (41) and the original transfer-function matrix $\mathbf{G}(s)$ as follows:

(i) square system ($p = q$)

$$\frac{d^j}{ds^j} \hat{\mathbf{G}}_M(s)|_{s=\sigma_i} = \frac{d^j}{ds^j} \mathbf{G}(s)|_{s=\sigma_i} \quad \text{for } j = 0, 1, \dots, \hat{\mu}_i - 1 \text{ and } i = 1, 2, \dots, l \quad (45a)$$

where l is the number of distinct points that appear in the ordered point system, and

$$(s_1, s_2, \dots, s_{2M}) = (\underbrace{\sigma_1, \dots, \sigma_1}_{\hat{\mu}_1}, \underbrace{\sigma_2, \dots, \sigma_2}_{\hat{\mu}_2}, \dots, \underbrace{\sigma_{l-1}, \dots, \sigma_{l-1}}_{\hat{\mu}_{l-1}}, \underbrace{\sigma_l, \dots, \sigma_l}_{\hat{\mu}_l}, \dots, \underbrace{\sigma_l, \dots, \sigma_l}_{\hat{\mu}_l}, \dots, \underbrace{\sigma_l, \dots, \sigma_l}_{\hat{\mu}_l}) \quad (45b)$$

which is obtained by truncating the last $2N - 2M$ points of the ordered point system in (4);

(ii) nonsquare system ($p \neq q$)

$$\frac{d^j}{ds^j} \hat{\mathbf{G}}_M(s)|_{s=\sigma_i} = \frac{d^j}{ds^j} \mathbf{G}(s)|_{s=\sigma_i} \quad \text{for } \bar{\mu}_i > 0, j = 0, 1, \dots, \bar{\mu}_i - 1 \text{ and } 1 \leq i \leq l \quad (46a)$$

where $\bar{\mu}_i$, the multiplicity of the point σ_i , appears in the ordered point system,

$$(s_2, s_4, \dots, s_{2M}) = (\underbrace{\sigma_1, \dots, \sigma_1}_{\bar{\mu}_1}, \underbrace{\sigma_2, \dots, \sigma_2}_{\bar{\mu}_2}, \dots, \underbrace{\sigma_l, \dots, \sigma_l}_{\bar{\mu}_l}) \quad (46b)$$

which is obtained from (45) by deleting the odd-indexed elements s_{2j-1} for $j = 1, 2, \dots, M$.

Notice from (46) that for a nonsquare system, the M th-degree reduced MFD obtained by the multipoint Cauer MCFE has the same Padé properties only around the even expansion points. However, since the multipoint Cauer MCFE involves the matrix pseudo-inverse in the calculation of odd matrix partial quotients, the obtained reduced-degree MFDs approximate the Padé properties around the odd

expansion points, i.e.

$$\frac{d^j}{ds^j} \hat{\mathbf{G}}_M(s)|_{s=\sigma_i} \approx \frac{d^j}{ds^j} \mathbf{G}(s)|_{s=\sigma_i}, \quad \text{for } j = 0, 1, \dots, \hat{\mu}_i - \bar{\mu}_i \text{ and } 1 \leq i \leq l \quad (47a)$$

where the expansion point σ_i appears $\hat{\mu}_i - \bar{\mu}_i (> 0)$ times in the ordered point system, and

$$(s_1, s_2, \dots, s_{2M-1}) = (\underbrace{\sigma_1, \dots, \sigma_1}_{\hat{\mu}_1 - \bar{\mu}_1}, \underbrace{\sigma_2, \dots, \sigma_2}_{\hat{\mu}_2 - \bar{\mu}_2}, \dots, \underbrace{\sigma_l, \dots, \sigma_l}_{\hat{\mu}_l - \bar{\mu}_l}) \quad (47b)$$

which is obtained from (43) by dropping the even-indexed elements s_{2j} for $j = 1, 2, \dots, M$.

IV. State-space Models Obtained from the Multipoint Cauer MCFE

4.1. State-space formulation of the multipoint Cauer MCFE

Let $\mathbf{Y}(s)$ and $\mathbf{U}(s)$ be, respectively, the Laplace transforms of the output and input vectors of the system. Then $\mathbf{Y}(s)$ is related to $\mathbf{U}(s)$ by

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s). \quad (48)$$

Premultiplying both sides of (48) by $\mathbf{G}^\#(s)$ and using (18), we have

$$\begin{aligned} \mathbf{U}(s) &= \mathbf{G}^\#(s)\mathbf{Y}(s) \\ &= [\mathbf{H}_1 + (s-s_1)[\mathbf{H}_2 + (s-s_2)\mathbf{G}_3(s)]^\# \mathbf{Y}(s) \\ &= \mathbf{H}_1 \mathbf{Y}(s) + \left[\frac{1}{(s-s_1)} (\mathbf{H}_2 + r_1 \mathbf{G}_3(s)) + \mathbf{G}_3(s) \right]^\# \mathbf{Y}(s) \end{aligned} \quad (49)$$

where

$$\mathbf{G}_i(s) = \left[\mathbf{H}_i + \left[\frac{1}{(s-s_i)} (\mathbf{H}_{i+1} + r_i \mathbf{G}_{i+2}(s)) + \mathbf{G}_{i+2}(s) \right]^\# \right]^\sim$$

and $r_i = s_i - s_{i+1}$ for $i = 1, 3, 5, \dots, 2N-1$. Let

$$\mathbf{X}_1(s) = \left[\frac{1}{s-s_1} (\mathbf{H}_2 + r_1 \mathbf{G}_3(s)) + \mathbf{G}_3(s) \right]^\# \mathbf{Y}(s). \quad (50)$$

Then, we have

$$\mathbf{Y}(s) = \left[\frac{1}{s-s_1} (\mathbf{H}_2 + r_1 \mathbf{G}_3(s)) + \mathbf{G}_3(s) \right] \mathbf{X}_1(s) \quad (51a)$$

and

$$\mathbf{X}_1(s) = \mathbf{U}(s) - \mathbf{H}_1 \mathbf{Y}(s). \quad (51b)$$

The above two relations allow one to construct the block diagram as shown in Fig. 1 for representing the relationship between $\mathbf{Y}(s)$ and $\mathbf{U}(s)$.

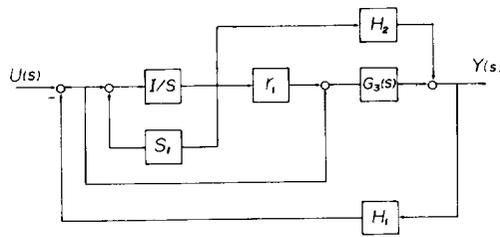


FIG. 1. Block diagram representation of $G_1(s)$.

Using a similar procedure to expand $G_3(s), G_5(s), \dots$, a realization block diagram (Fig. 2) for $G(s)$ can be achieved. It is readily apparent from Fig. 2 that a canonical state-space representation of $G(s)$ can be written as follows:

$$\dot{z} = \mathbf{H}z + \mathbf{G}u \tag{52a}$$

$$y = \mathbf{F}z \tag{52b}$$

with

$$z = [z_1^T \ z_2^T \ \dots \ z_N^T]^T, \quad z_i = [z_{i,1} \ z_{i,2} \ \dots \ z_{i,q}]^T$$

$$\mathbf{H} = -\mathbf{H}_0\mathbf{H}_c + \mathbf{R} \otimes \mathbf{I}_q$$

$$\mathbf{H}_0 = \begin{bmatrix} \mathbf{H}_1 & & & \\ \mathbf{H}_1 & \mathbf{H}_3 & & \mathbf{O} \\ \vdots & \vdots & \ddots & \\ \mathbf{H}_1 & \mathbf{H}_3 & \dots & \mathbf{H}_{2N-1} \end{bmatrix}, \quad \mathbf{H}_c = \begin{bmatrix} \mathbf{H}_2 & \mathbf{H}_4 & \dots & \mathbf{H}_{2N} \\ & \mathbf{H}_4 & \dots & \mathbf{H}_{2N} \\ \mathbf{O} & & \ddots & \vdots \\ & & & \mathbf{H}_{2N} \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} s_1 & & & & & \\ s_1 & r_3 & & & & \mathbf{O} \\ s_1 & s_3 & r_5 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ s_1 & s_3 & s_5 & \dots & r_{2N-3} & \\ s_1 & s_3 & s_5 & \dots & s_{2N-3} & r_{2N-1} \end{bmatrix}$$

$$\mathbf{G} = \underbrace{[\mathbf{I}_q \ \mathbf{I}_q \ \dots \ \mathbf{I}_q]^T}_{N \text{ block elements}}$$

$$\mathbf{F} = [\mathbf{H}_2 \ \mathbf{H}_4 \ \dots \ \mathbf{H}_{2N}]$$

where the superscript T designates the transpose of a matrix or vector, and \otimes denotes the Kronecker product (16).

4.2. Link between MCFE and block companion forms

Let $\mathbf{A}_{1,N} = \mathbf{I}_q$, then a controllable block companion form for the realization of the MFD (2) can be written as follows (17):

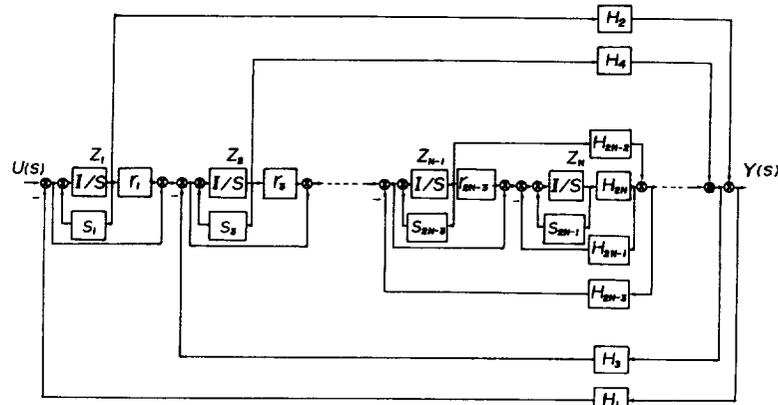


FIG. 2. Realization block diagram of multipoint Cauer matrix CFE.

$$\dot{x} = Ax + Bu \tag{53a}$$

$$y = Cx \tag{53b}$$

where

$$x = [x_1^T \ x_2^T \ \dots \ x_N^T]^T, \quad x_i = [x_{i,1} \ x_{i,2} \ \dots \ x_{i,q}]^T$$

$$A = \begin{bmatrix} \mathbf{O}_q & \mathbf{I}_q & & & \\ \mathbf{O}_q & \mathbf{O}_q & \mathbf{I}_q & & \mathbf{O} \\ \vdots & \vdots & \vdots & \ddots & \\ \mathbf{O}_q & \mathbf{O}_q & \mathbf{O}_q & \dots & \mathbf{I}_q \\ -\mathbf{A}_{1,0} & -\mathbf{A}_{1,1} & -\mathbf{A}_{1,2} & \dots & -\mathbf{A}_{1,N-1} \end{bmatrix}$$

$$B = [\mathbf{O}_q \ \mathbf{O}_q \ \dots \ \mathbf{I}_q]^T$$

$$C = [\mathbf{A}_{2,0} \ \mathbf{A}_{2,1} \ \dots \ \mathbf{A}_{2,N-1}]$$

Assume that the system is completely controllable with controllability index equal to Nq . Then there exists a nonsingular block transformation matrix P which links the block state vectors z and x by

$$z = Px \tag{54}$$

and satisfies the following relations:

$$PA = HP \tag{55a}$$

$$PB = G \tag{55b}$$

$$C = FP. \tag{55c}$$

The desired block similarity transformation matrix P can be derived from the relation (18a) and (55c). Let the block similarity transformation matrix P be of

the form

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \cdots & \mathbf{P}_{1,N} \\ \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \cdots & \mathbf{P}_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}_{N,1} & \mathbf{P}_{N,2} & \cdots & \mathbf{P}_{N,N} \end{bmatrix} \quad (56)$$

where the block elements $\mathbf{P}_{i,j}$ are $q \times q$ matrices. Define the matrix polynomials

$$\mathbf{P}_i(s) = \sum_{j=1}^N \mathbf{P}_{i,j} s^{j-1}, \quad i = 1, 2, \dots, N. \quad (57)$$

Then the relation (55c) is equivalent to

$$\mathbf{A}_2(s) = \mathbf{H}_2 \mathbf{P}_1(s) + \mathbf{H}_4 \mathbf{P}_2(s) + \cdots + \mathbf{H}_{2N} \mathbf{P}_N(s). \quad (58)$$

Rewrite the recursion in (18b) as

$$\mathbf{A}_i(s) = (s - s_i) \mathbf{A}_{i+2}(s) + \mathbf{H}_i \mathbf{A}_{i+1}(s), \quad i = 1, 2, \dots, 2N. \quad (59)$$

Then, by applying the above relations recursively for $i = 2, 4, \dots, 2N$ and noting that $\mathbf{A}_{2N+2}(s) = \mathbf{O}_{p \times q}$, we have

$$\mathbf{A}_2(s) = \mathbf{H}_2 \mathbf{A}_3(s) + (s - s_2) \mathbf{H}_4 \mathbf{A}_5(s) + (s - s_2)(s - s_4) \mathbf{H}_6 \mathbf{A}_7(s) + \cdots + (s - s_2) \cdots (s - s_{2N-2}) \mathbf{H}_{2N} \mathbf{A}_{2N+1}(s). \quad (60)$$

Comparing (58) and (60), we obtain

$$\mathbf{P}_i(s) = \prod_{j=1}^{i-1} (s - s_{2j}) \mathbf{A}_{2i+1}(s), \quad i = 1, 2, \dots, N. \quad (61)$$

Using (57) and equating the coefficients of the like powers of s^j in (61), and using the relations in (57), we have the explicit form for $\mathbf{P}_{i,j}$ as

$$\mathbf{P}_{i,j} = \sum_{k=1}^j r_{i,k} \mathbf{A}_{2i+1, j-k+1}, \quad i = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, n \quad (62a)$$

where $\mathbf{A}_{i,j}$ are the coefficients of the matrix polynomials defined in (18b, c), and $r_{i,k}$ are given by

$$r_{i,k} = \begin{cases} 0 & \text{for } i < k \text{ or } k = 0 \\ 1 & \text{for } i = k \\ r_{i-1, k-1} - s_{i-2} r_{i-1, k} & \text{for } i > k. \end{cases} \quad (62b)$$

4.3. Reduced-dimensional state-space models

Truncation of the last $(2N - 2M)$ matrix partial quotients of the multipoint Cauer MCFE in (18) after the first $2M$ matrix partial quotients is equivalent to discarding the $N - M$ inner block loops of Fig. 2. Hence, the multipoint Cauer MCFE canonical state-space model corresponding to the M th-degree $\hat{\mathbf{G}}_M(s)$ can

be readily obtained as

$$\dot{\hat{\mathbf{z}}} = \hat{\mathbf{H}}\hat{\mathbf{z}} + \hat{\mathbf{G}}\mathbf{u} \quad (63a)$$

$$\hat{\mathbf{y}} = \hat{\mathbf{F}}\hat{\mathbf{z}} \quad (63b)$$

where

$$\begin{aligned} \hat{\mathbf{z}} &= [\hat{\mathbf{z}}_1^T \quad \hat{\mathbf{z}}_2^T \quad \dots \quad \hat{\mathbf{z}}_M^T]^T, \quad \hat{\mathbf{z}}_i = [\hat{z}_{i,1} \quad \hat{z}_{i,2} \quad \dots \quad \hat{z}_{i,q}]^T \\ \hat{\mathbf{H}} &= -\hat{\mathbf{H}}_0\hat{\mathbf{H}}_e + \hat{\mathbf{R}} \otimes \mathbf{I}_q \\ \hat{\mathbf{H}}_0 &= \begin{bmatrix} \mathbf{H}_1 & & & \\ \mathbf{H}_1 & \mathbf{H}_3 & & \mathbf{O} \\ \vdots & \vdots & \ddots & \\ \mathbf{H}_1 & \mathbf{H}_3 & \dots & \mathbf{H}_{2M-1} \end{bmatrix}, \quad \hat{\mathbf{H}}_e = \begin{bmatrix} \mathbf{H}_2 & \mathbf{H}_4 & \dots & \mathbf{H}_{2M} \\ & \mathbf{H}_4 & \dots & \mathbf{H}_{2M} \\ \mathbf{O} & & \ddots & \vdots \\ & & & \mathbf{H}_{2M} \end{bmatrix} \\ \hat{\mathbf{R}} &= \begin{bmatrix} s_1 & & & & & \\ s_1 & r_3 & & & & \mathbf{O} \\ s_1 & s_3 & r_5 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ s_1 & s_3 & s_5 & \dots & r_{2M-3} & \\ s_1 & s_3 & s_5 & \dots & s_{2M-3} & r_{2M-1} \end{bmatrix} \\ \hat{\mathbf{G}} &= \underbrace{[\mathbf{I}_q \quad \mathbf{I}_q \quad \dots \quad \mathbf{I}_q]^T}_{M \text{ block elements}} \\ \hat{\mathbf{F}} &= [\mathbf{H}_2 \quad \mathbf{H}_4 \quad \dots \quad \mathbf{H}_{2M}]. \end{aligned}$$

It is obvious that $\hat{\mathbf{H}}$ is the upper-left M by M block submatrix of \mathbf{H} , and $\hat{\mathbf{G}}$ and $\hat{\mathbf{F}}$ are the upper block subvectors of \mathbf{G} and \mathbf{F} , respectively. Hence, the truncation procedure in the time domain can be represented by the following mathematical relations:

$$\hat{\mathbf{H}} = \mathbf{J}\mathbf{H}\mathbf{J}^T \quad (64a)$$

$$\hat{\mathbf{G}} = \mathbf{J}\mathbf{G} \quad (64b)$$

$$\hat{\mathbf{F}} = \mathbf{F}\mathbf{J}^T \quad (64c)$$

where

$$\mathbf{J} = [\mathbf{I}_{Mq} \quad \mathbf{O}_{Mq \times (N-M)q}].$$

Based on the multi-feedback multi-feedforward control theory (2, 10) and the realization block diagram of Fig. 2, we know that the inner block states have fewer contributions to the system output. Hence, it is reasonable to consider that for an appropriate M , the following approximation holds

$$\hat{\mathbf{z}} \approx \mathbf{J}\mathbf{z}. \quad (65)$$

With this approximation and the relation (64), we conclude that the state vector $\hat{\mathbf{z}}$ of the reduced-dimensional state-space model is approximately related to that of the original system in (53) by

$$\dot{\mathbf{z}} = \mathbf{Q}\mathbf{x} \quad (66a)$$

where

$$\mathbf{Q} = \mathbf{J}\mathbf{P}. \quad (66b)$$

The \mathbf{Q} is called an approximate aggregation matrix. The availability of this approximate aggregation matrix enables the reduced-dimensional state-space models obtained by the multipoint Cauer MCFE method to be useful in designing reduced-order controllers.

V. An Illustrative Example

To illustrate the proposed algorithms, we consider a two-input–three-output system characterized by its third-degree MFD

$$\begin{aligned} \mathbf{G}(s) = & \left[\begin{array}{cc} -7.6051 & -18.4986 \\ -35.0581 & -85.4861 \\ -25.4654 & -62.0624 \end{array} \right] + \left[\begin{array}{cc} -23.5164 & -49.8596 \\ -4.5654 & 21.8066 \\ -6.4341 & 8.5709 \end{array} \right] s \\ & + \left[\begin{array}{cc} -6.6984 & 3.8209 \\ 1.8038 & -0.8574 \\ 0.2685 & 0.0243 \end{array} \right] s^2 \left[\begin{array}{cc} -1.2972 & -2.5216 \\ -85.9888 & -209.6110 \end{array} \right] \\ & + \left[\begin{array}{cc} -64.1796 & -148.9154 \\ -167.1634 & -326.1264 \end{array} \right] s + \left[\begin{array}{cc} -16.0738 & 29.8019 \\ -39.1889 & 40.4941 \end{array} \right] s^2 \\ & + \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] s^3 \end{aligned} .$$

The transfer-function matrix is given by:

$$\mathbf{G}(s) = \begin{bmatrix} g_{1,1}(s) & g_{1,2}(s) \\ g_{2,1}(s) & g_{2,2}(s) \\ g_{3,1}(s) & g_{3,2}(s) \end{bmatrix}$$

where

$$\begin{aligned} g_{1,1}(s) &= (3.440200 + 29.863961s + 34.330077s^2 - 90.583719s^3 \\ &\quad - 145.025211s^4 - 6.6984s^5)/d(s) \\ g_{2,1}(s) &= (-2.283757 - 24.693759s - 87.412685s^2 - 96.945897s^3 \\ &\quad + 34.877295s^4 + 1.8038s^5)/d(s) \\ g_{3,1}(s) &= (1.156658 + 16.036971s + 13.523988s^2 - 33.627214s^3 \\ &\quad + 5.390856s^4 + 0.2685s^5)/d(s) \end{aligned}$$

$$g_{1,2}(s) = (4.819364 + 60.095159s + 200.157141s^2 + 241.049291s^3 + 88.349065s^4 + 3.8209s^5)/d(s)$$

$$g_{2,2}(s) = (22.490064 + 225.973085s + 345.147913s^2 + 23.697755s^3 - 18.168391s^4 - 0.8574s^5)/d(s)$$

$$g_{3,2}(s) = (16.293793 + 163.607382s + 248.927928s^2 + 30.343293s^3 + 0.178497s^4 + 0.0243s^5)/d(s)$$

$$d(s) = 55.078031 + 649.225525s + 1817.984559s^2 + 1578.243399s^3 + 126.703614s^4 + 24.4203s^5 + s^6.$$

The controllable block companion form corresponding to this MFD is

$$\begin{bmatrix} \dot{x}_{1,1} \\ \dot{x}_{1,2} \\ \dot{x}_{2,1} \\ \dot{x}_{2,2} \\ \dot{x}_{3,1} \\ \dot{x}_{3,2} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1.2972 & 2.5216 \\ 85.9888 & 209.6110 \end{bmatrix} & \begin{bmatrix} 64.1796 & 148.9154 \\ 167.1634 & 326.1264 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \\ \begin{bmatrix} 16.0738 & -29.8019 \\ 39.1889 & -40.4941 \end{bmatrix} & \end{bmatrix} \begin{bmatrix} x_{1,1} \\ x_{1,2} \\ x_{2,1} \\ x_{2,2} \\ x_{3,1} \\ x_{3,2} \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -7.6051 & -18.4986 \\ -35.0581 & -85.4861 \\ -25.4654 & -62.0624 \end{bmatrix} & \begin{bmatrix} -23.5164 & -49.8596 \\ -4.5654 & 21.8066 \\ -6.4341 & 8.5709 \end{bmatrix} \\ \begin{bmatrix} -6.6984 & 3.8209 \\ 1.8038 & -0.8574 \\ 0.2685 & 0.0243 \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_{1,1} \\ x_{1,2} \\ x_{2,1} \\ x_{2,2} \\ x_{3,1} \\ x_{3,2} \end{bmatrix}$$

The Taylor series expansions of $G(s)$ around expansion points at $s = 0, 1, 2$ are evaluated as

$$\begin{aligned}
 \mathbf{G}(s) &= \begin{bmatrix} 0.062460 & 0.087501 \\ -0.041464 & 0.408331 \\ 0.021000 & 0.295831 \end{bmatrix} + \begin{bmatrix} -0.194033 & 0.059688 \\ 0.040411 & -0.710370 \\ 0.043629 & -0.516607 \end{bmatrix} s \\
 &+ \begin{bmatrix} 0.848784 & 0.042327 \\ -0.694785 & 1.161968 \\ -0.961898 & 0.844367 \end{bmatrix} s^2 + \dots \\
 &= \begin{bmatrix} -0.041074 & 0.140686 \\ -0.041070 & 0.140685 \\ 0.000647 & 0.108021 \end{bmatrix} + \begin{bmatrix} -0.091768 & 0.046501 \\ 0.012861 & -0.105319 \\ -0.009672 & -0.068097 \end{bmatrix} (s-1) \\
 &+ \begin{bmatrix} 0.005283 & -0.005326 \\ 0.008434 & 0.058837 \\ 0.008503 & 0.042513 \end{bmatrix} (s-1)^2 + \dots \\
 &= \begin{bmatrix} -0.126800 & 0.181971 \\ -0.023260 & 0.071566 \\ -0.003600 & 0.065727 \end{bmatrix} + \begin{bmatrix} -0.078945 & 0.036070 \\ 0.020381 & -0.046636 \\ -0.000845 & -0.026606 \end{bmatrix} (s-2) \\
 &+ \begin{bmatrix} 0.007431 & -0.005244 \\ 0.000944 & 0.014180 \\ 0.002074 & 0.009571 \end{bmatrix} (s-2)^2 + \dots
 \end{aligned}$$

Choosing the ordered expansion point system as $(s_1, s_1, s_2, s_2, s_3, s_3) = (0, 0, 1, 1, 2, 2)$ and using the recursive algorithms (18) and (33), we construct the expansion arrays as shown in Tables IV and V for evaluating the multipoint Cauer MCFE from the MFD and Taylor series expansion of $\mathbf{G}(s)$, respectively.

We obtain from (18a) and (34), the same set matrix of partial quotients as follows :

$$\begin{aligned}
 \mathbf{H}_1 &= \begin{bmatrix} 10.781239 & -5.586419 & 4.521980 \\ 0.550334 & 1.447022 & 1.220228 \end{bmatrix} & \mathbf{H}_2 &= \begin{bmatrix} 0.029679 & 0.095386 \\ -0.019033 & 0.224562 \\ 0.010361 & 0.196506 \end{bmatrix} \\
 \mathbf{H}_3 &= \begin{bmatrix} -10.576133 & -5.571830 & -4.021421 \\ 4.017897 & -8.535413 & -4.024899 \end{bmatrix} & \mathbf{H}_4 &= \begin{bmatrix} -6.737767 & 3.651197 \\ 1.817531 & -1.093386 \\ 0.235814 & -0.184328 \end{bmatrix} \\
 \mathbf{H}_5 &= \begin{bmatrix} 165.364859 & 70.422714 & 271.800239 \\ 209.425455 & 49.768519 & 129.772792 \end{bmatrix} & \mathbf{H}_6 &= \begin{bmatrix} 0.009688 & 0.074317 \\ 0.005302 & 0.011424 \\ 0.022325 & 0.012122 \end{bmatrix} .
 \end{aligned}$$

TABLE IV
Expansion array for evaluating $A_{i,j}$

$\begin{bmatrix} -1.2972 & -2.5216 \\ -85.9888 & -209.6110 \end{bmatrix}$	$\begin{bmatrix} -64.1796 & -148.9154 \\ -167.1634 & -326.1264 \end{bmatrix}$	$\begin{bmatrix} -16.0738 & 29.8019 \\ -39.1889 & 40.4941 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$\begin{bmatrix} -7.6051 & -18.4986 \\ -35.0581 & -85.4861 \\ -25.4654 & -62.0624 \end{bmatrix}$	$\begin{bmatrix} -23.5164 & -49.8596 \\ -4.5654 & 21.8066 \\ -6.4341 & 8.5709 \end{bmatrix}$	$\begin{bmatrix} -6.6984 & 3.8209 \\ 1.8038 & -0.8574 \\ 0.2685 & 0.0243 \end{bmatrix}$	
$\begin{bmatrix} 192.946970 & 471.696242 \\ -139.764214 & -340.700047 \end{bmatrix}$	$\begin{bmatrix} 65.005884 & -16.291817 \\ -38.440311 & 39.602353 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	
$\begin{bmatrix} -21.779035 & -53.153589 \\ 5.304092 & 12.603329 \\ 0.446132 & 0.957600 \end{bmatrix}$	$\begin{bmatrix} -6.728079 & 3.725514 \\ 1.822833 & -1.081962 \\ 0.258139 & -0.172206 \end{bmatrix}$		
$\begin{bmatrix} 6.043425 & 16.388694 \\ 5.190035 & 15.705515 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		
$\begin{bmatrix} 0.009688 & 0.074317 \\ 0.005302 & 0.011424 \\ 0.022325 & 0.012122 \end{bmatrix}$			
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$			

TABLE Va
Expansion array for evaluating $G_{1,j,k}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\begin{bmatrix} 0.062460 & 0.087501 \\ -0.041464 & 0.408331 \\ 0.021000 & 0.295831 \end{bmatrix}$	$\begin{bmatrix} -0.194033 & 0.059688 \\ 0.040411 & -0.710370 \\ 0.043629 & -0.516607 \end{bmatrix}$
$\begin{bmatrix} 2.120382 & -2.275849 \\ -0.004929 & 1.625451 \end{bmatrix}$	

TABLE Vb
Expansion array for evaluating $G_{2,j,k}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\begin{bmatrix} -0.041074 & 0.140686 \\ -0.041070 & 0.140685 \\ 0.000647 & 0.108021 \end{bmatrix}$	$\begin{bmatrix} -0.091768 & 0.046501 \\ 0.012861 & -0.105319 \\ -0.009672 & -0.068097 \end{bmatrix}$
$\begin{bmatrix} 1.210471 & -1.219319 \\ 0.081244 & 0.587192 \end{bmatrix}$	$\begin{bmatrix} -0.105514 & 0.437559 \\ -0.037548 & -0.377289 \end{bmatrix}$
$\begin{bmatrix} -0.084749 & 0.120865 \\ -0.036275 & -0.014384 \\ -0.027860 & 0.005267 \end{bmatrix}$	$\begin{bmatrix} -0.000306 & -0.051362 \\ 0.055560 & 0.002118 \\ 0.026659 & -0.003759 \end{bmatrix}$
$\begin{bmatrix} 0.308025 & -0.108965 \\ 0.545205 & -0.167977 \end{bmatrix}$	

TABLE Vc
Expansion array for evaluating $G_{3,j,k}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\begin{bmatrix} -0.126800 & 0.181971 \\ -0.023260 & 0.071566 \\ -0.003600 & 0.065727 \end{bmatrix}$	$\begin{bmatrix} -0.078945 & 0.036070 \\ 0.020381 & -0.046636 \\ -0.000845 & -0.026606 \end{bmatrix}$
$\begin{bmatrix} 1.126698 & -0.929643 \\ 0.053916 & 0.358048 \end{bmatrix}$	$\begin{bmatrix} -0.078950 & 0.200273 \\ -0.019465 & -0.138975 \end{bmatrix}$
$\begin{bmatrix} -0.082691 & 0.087704 \\ -0.006962 & -0.013266 \\ -0.012934 & 0.002500 \end{bmatrix}$	$\begin{bmatrix} 0.003973 & -0.022161 \\ 0.015105 & 0.000825 \\ 0.008366 & -0.001936 \end{bmatrix}$
$\begin{bmatrix} 0.161345 & -0.065929 \\ 0.274680 & -0.097505 \end{bmatrix}$	$\begin{bmatrix} -0.080466 & 0.028638 \\ -0.147505 & 0.046821 \end{bmatrix}$
$\begin{bmatrix} 0.001500 & -0.000500 \\ 0.000121 & -0.000048 \\ -0.000350 & 0.000074 \end{bmatrix}$	$\begin{bmatrix} -0.001119 & 0.000338 \\ -0.000046 & 0.000017 \\ 0.000502 & -0.000133 \end{bmatrix}$
$\begin{bmatrix} -0.028680 & 0.007509 \\ 0.023921 & -0.007708 \end{bmatrix}$	

Once the matrix partial quotients H_i are evaluated, the canonical block state-space model corresponding to the multipoint Cauer MCFE can be constructed according to (52) as follows:

$$\begin{bmatrix} \dot{z}_{1,1} \\ \dot{z}_{1,2} \\ \dot{z}_{2,1} \\ \dot{z}_{2,2} \\ \dot{z}_{3,1} \\ \dot{z}_{3,2} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -0.473153 & -0.662478 \\ -0.001435 & -0.617223 \end{bmatrix} \\ \begin{bmatrix} -0.473153 & -0.662478 \\ -0.001435 & -0.617223 \end{bmatrix} \\ \begin{bmatrix} -0.473153 & -0.662478 \\ -0.001435 & -0.617223 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 81.728620 & -44.639017 \\ 0.790269 & -0.202302 \end{bmatrix} \\ \begin{bmatrix} 22.544374 & -12.856893 \\ 44.324429 & -23.946844 \end{bmatrix} \\ \begin{bmatrix} 21.544374 & -12.856893 \\ 44.324429 & -24.946844 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} -0.175783 & -0.792222 \\ -0.040246 & -0.072222 \end{bmatrix} \\ \begin{bmatrix} 0.046004 & 0.106164 \\ 0.055941 & -0.224518 \end{bmatrix} \\ \begin{bmatrix} -5.997421 & -16.282529 \\ -5.134094 & -15.930034 \end{bmatrix} \end{bmatrix} \begin{bmatrix} z_{1,1} \\ z_{1,2} \\ z_{2,1} \\ z_{2,2} \\ z_{3,1} \\ z_{3,2} \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0.029679 & 0.095386 \\ -0.019033 & 0.224562 \\ 0.010361 & 0.196506 \end{bmatrix} & \begin{bmatrix} -6.737767 & 3.651197 \\ 1.817531 & -1.093386 \\ 0.235814 & -0.184328 \end{bmatrix} \\ \begin{bmatrix} 0.009688 & 0.074317 \\ 0.005302 & 0.011424 \\ 0.022325 & 0.012122 \end{bmatrix} & \begin{bmatrix} z_{1,1} \\ z_{1,2} \\ z_{2,1} \\ z_{2,2} \\ z_{3,1} \\ z_{3,2} \end{bmatrix} \end{bmatrix}$$

Using (62) with $A_{i,j}$ extracted from Table IV, we obtain the block similarity transformation matrix for relating the block state vectors z and x in (54) as follows :

$$P = \begin{bmatrix} \begin{bmatrix} 192.946970 & 471.696242 \\ -139.764214 & -340.700047 \end{bmatrix} & \begin{bmatrix} 65.005884 & -16.291817 \\ -38.440311 & 39.602353 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 6.043425 & 16.388694 \\ 5.190035 & 15.705515 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

Based on the model reduction method proposed in this paper, the second-degree reduced MFD model, $\hat{G}_2(s)$, derived from the matrix partial quotients, H_i , $i = 1, 2, 3, 4$ is formed as

$$G_2(s) = \begin{bmatrix} \begin{bmatrix} -2.396027 & 1.321639 \\ -10.902575 & 5.937033 \\ -7.941521 & 4.336653 \end{bmatrix} & \begin{bmatrix} -6.708088 & 3.746583 \\ 1.798498 & -0.868824 \\ 0.246175 & 0.012178 \end{bmatrix} \end{bmatrix} s$$

$$\begin{bmatrix} \begin{bmatrix} -0.837195 & 0.692416 \\ -26.785352 & 14.610068 \end{bmatrix} & \begin{bmatrix} -22.071221 & 13.519371 \\ -44.322994 & 24.564066 \end{bmatrix} \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 \Big)^{-1} \\ = \begin{bmatrix} \hat{g}_{1,1}(s) & \hat{g}_{1,2}(s) & \hat{g}_{1,3}(s) \\ \hat{g}_{2,1}(s) & \hat{g}_{2,2}(s) & \hat{g}_{2,3}(s) \end{bmatrix}$$

where

$$\begin{aligned} \hat{g}_{1,1}(s) &= (0.394446 + 2.070750s - 1.114170s^2 - 6.708088s^3) / \hat{d}(s) \\ \hat{g}_{1,2}(s) &= (0.552578 + 4.730760s + 9.319107s^2 + 3.746583s^3) / \hat{d}(s) \\ \hat{g}_{2,1}(s) &= (-0.261851 - 1.660099s - 5.233055s^2 + 1.798498s^3) / \hat{d}(s) \end{aligned}$$

$$\hat{g}_{2,2}(s) = (2.578663 + 15.840457s + 0.798490s^2 - 0.868824s^3)/\hat{d}(s)$$

$$\hat{g}_{3,1}(s) = (0.132620 + 1.060232s - 1.354695s^2 + 0.246175s^3)/\hat{d}(s)$$

$$\hat{g}_{3,2}(s) = (1.868212 + 11.468484s + 0.739738s^2 + 0.012178s^3)/\hat{d}(s)$$

$$\hat{d}(s) = 6.315131 + 49.784100s + 70.832926s^2 + 2.492846s^3 + s^4.$$

Furthermore, the multipoint Cauer MCFE state-space model corresponding to $\hat{\mathbf{G}}_2(s)$ can be readily obtained as

$$\begin{bmatrix} \dot{z}_{1,1} \\ \dot{z}_{1,2} \\ \dot{z}_{2,1} \\ \dot{z}_{2,2} \end{bmatrix} = \begin{bmatrix} -0.473153 & -0.662478 \\ -0.001435 & -0.617223 \\ -0.473153 & -0.662478 \\ -0.001435 & -0.617223 \end{bmatrix} \begin{bmatrix} z_{1,1} & -44.639017 \\ z_{1,2} & -0.202302 \\ z_{2,1} & -12.856893 \\ z_{2,2} & -23.946844 \end{bmatrix}$$

$$\cdot \begin{bmatrix} z_{1,1} \\ z_{1,2} \\ z_{2,1} \\ z_{2,2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.029679 & 0.095386 \\ -0.019033 & 0.224562 \\ 0.010361 & 0.196506 \end{bmatrix} \begin{bmatrix} -6.737767 & 3.651197 \\ 1.817531 & -1.093386 \\ 0.235814 & -0.184328 \end{bmatrix} \begin{bmatrix} z_{1,1} \\ z_{1,2} \\ z_{2,1} \\ z_{2,2} \end{bmatrix}.$$

The Taylor series expansions of $\hat{\mathbf{G}}_2(s)$ around $s = 0, 1, 2$ are, respectively, evaluated as

$$\hat{\mathbf{G}}_2(s) = \begin{bmatrix} 0.062460 & 0.087501 \\ -0.041464 & 0.408331 \\ 0.021000 & 0.295831 \end{bmatrix} + \begin{bmatrix} -0.164492 & 0.059321 \\ 0.063997 & -0.710663 \\ 0.002335 & -0.516094 \end{bmatrix} s$$

$$+ \begin{bmatrix} 0.419732 & 0.026594 \\ -0.868085 & 1.148819 \\ -0.468475 & 0.867507 \end{bmatrix} s^2 + \dots$$

$$= \begin{bmatrix} -0.041074 & 0.140686 \\ -0.041070 & 0.140685 \\ 0.000647 & 0.108021 \end{bmatrix} + \begin{bmatrix} -0.091599 & 0.046459 \\ 0.012294 & -0.105179 \\ -0.007988 & -0.068514 \end{bmatrix} (s-1)$$

$$+ \begin{bmatrix} 0.006230 & -0.005601 \\ 0.008666 & 0.058842 \\ 0.007286 & 0.042725 \end{bmatrix} (s-1)^2 + \dots$$

$$= \begin{bmatrix} -0.126037 & 0.181728 \\ -0.023818 & 0.071745 \\ -0.002814 & 0.065532 \end{bmatrix} + \begin{bmatrix} -0.077841 & 0.035700 \\ 0.019754 & -0.046413 \\ -0.000689 & -0.026610 \end{bmatrix} (s-2) \\ + \begin{bmatrix} 0.007584 & -0.005306 \\ 0.000869 & 0.014214 \\ 0.001655 & 0.009714 \end{bmatrix} (s-2)^2 + \dots$$

Comparing the above series with those of $\mathbf{G}(s)$, we can verify that the partial matrix Padé properties of $\hat{\mathbf{G}}(s)$ given in (46a) and (47a) match those of $\mathbf{G}(s)$.

For comparison of the time responses between reduced-degree models and $\mathbf{G}(s)$, the second-degree reduced MFD obtained by the single-point Caue second form of the MCFE (12) is given below :

$$\hat{\mathbf{G}}_2^*(s) = \begin{bmatrix} \begin{bmatrix} -2.446550 & 1.375684 \\ -11.180131 & 6.225057 \\ -8.136202 & 4.539832 \end{bmatrix} + \begin{bmatrix} -6.893735 & 3.936407 \\ 1.825865 & -0.899738 \\ 0.213142 & 0.045311 \end{bmatrix} s \\ \cdot \begin{bmatrix} \begin{bmatrix} -0.711690 & 0.584829 \\ -27.452343 & 15.304513 \end{bmatrix} + \begin{bmatrix} -22.590895 & 14.079862 \\ -45.510647 & 25.793547 \end{bmatrix} s \\ + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 \end{bmatrix}^{-1} = \begin{bmatrix} \hat{g}_{1,1}^*(s) & \hat{g}_{1,2}^*(s) & \hat{g}_{1,3}^*(s) \\ \hat{g}_{2,1}^*(s) & \hat{g}_{2,2}^*(s) & \hat{g}_{2,3}^*(s) \end{bmatrix}$$

where

$$\hat{g}_{1,1}^*(s) = (0.322474 + 2.061388s - 1.111976s^2 - 6.893375s^3) / \hat{d}^*(s) \\ \hat{g}_{1,2}^*(s) = (0.451753 + 4.599323s + 9.511552s^2 + 3.936407s^3) / \hat{d}^*(s) \\ \hat{g}_{2,1}^*(s) = (-0.214073 - 1.824823s - 5.032249s^2 + 1.825865s^3) / \hat{d}^*(s) \\ \hat{g}_{2,2}^*(s) = (2.108154 + 16.357619s + 0.843013s^2 - 0.899738s^3) / \hat{d}^*(s) \\ \hat{g}_{3,1}^*(s) = (0.108422 + 1.255139s - 0.576348s^2 + 0.213142s^3) / \hat{d}^*(s) \\ \hat{g}_{3,2}^*(s) = (1.527333 + 11.840828s + 0.515189s^2 + 0.045311s^3) / \hat{d}^*(s) \\ \hat{d}^*(s) = 5.162856 + 49.041488s + 72.677136s^2 + 3.202651s^3 + s^4.$$

The values of the impulse response energy (18) for transfer function elements and outputs of the original system and those of the reduced-degree MFD are listed in Table VI. The values of the integral squared-error of the impulse and the unit-step time responses for each output of the reduced models are shown in Table VII. The above values are usually taken as criteria for choosing proper reduced-degree models. The unit-step time responses for each output of the original system $\mathbf{G}(s)$ and the reduced-degree models $\hat{\mathbf{G}}_2(s)$ are plotted in Fig. 3. It is observed that the time-responses of the reduced model are close to those of the original system.

TABLE VI
The values of impulse response energy of the original and reduced-degree models

	$\hat{G}_1^*(s)$	$\hat{G}_2(s)$	$G(s)$
$g_{1,1}(s)$	9.493938	12.707962	14.771495
$g_{2,1}(s)$	0.748100	1.037583	1.283513
$g_{3,1}(s)$	0.008769	0.022492	0.042638
$g_{1,2}(s)$	3.219912	4.134660	4.683292
$g_{2,2}(s)$	0.298879	0.382659	0.448826
$g_{3,2}(s)$	0.027614	0.032275	0.037869
$y_1(s)$	2.039829	2.888713	3.421429
$y_2(s)$	0.191937	0.267885	0.331350
$y_3(s)$	0.030720	0.032574	0.038061

TABLE VII
The integral squared-errors of the impulse and unit-step responses between the original and reduced-degree models

	Impulse response			Unit-step response		
	$y_1(s)$	$y_2(s)$	$y_3(s)$	$y_1(s)$	$y_2(s)$	$y_3(s)$
$\hat{G}_1^*(s)$	0.467087	0.045071	0.003970	0.006885	0.000700	0.000136
$\hat{G}_2(s)$	0.135860	0.014271	0.001245	0.002030	0.000257	0.000098

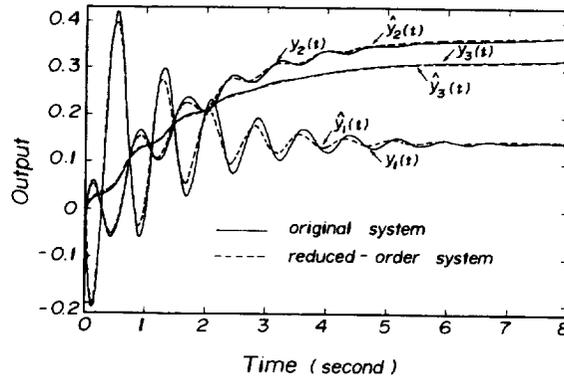


FIG. 3. Unit-step responses of the original and reduced-degree models.

VI. Conclusions

A matrix continued-fraction methodology has been established for modeling and model reduction of multivariable systems having an unequal number of inputs and outputs. The developed methodology is based on the multipoint Caueer matrix continued-fraction expansion and the matrix pseudo-inverse, and is applicable to obtain both the frequency-domain reduced-degree MFDs and time-domain reduced-dimensional state-space models. The connection of the presented multipoint Caueer MCFE method involving matrix pseudo-inverse to the multipoint matrix Padé approximation method is also established. Although the algorithms in the paper are derived for a right MFD, the extension to a left MFD is straightforward.

The use of a multipoint continued fraction expansion gives good reduced-order models that provide satisfactory approximation of the original system not only in the transient response but also in the steady-state response. Although the stability of the reduced models by the multipoint Caueer MCFE is not completely guaranteed, the flexibility of choosing expansion points provides the highest probability for finding stable reduced-order models (19).

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